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The Distributive Hull of a Ring

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Let R be a commutative ring with identity. An extension $M \subseteq N$ of R-modules is said to be distributive if it satisfies the following condition:

$$M \cap (X + Y) = (M \cap X) + (M \cap Y), \qquad \text{for all submodules } X, Y \text{ of } N.$$

In [2], Davison has shown that every R-module M which is locally non-zero at every maximal ideal of R has a maximal distributive extension and has raised the question: Is this unique up to M-isomorphism, in which case one can denote it by $D(M)$ and call it the distributive hull of M [1, 5].

In this paper we answer the question in the affirmative in the case when M is the R-module R, and we show that $D(R)$ is a ring contained in the maximal quotient ring $Q(R)$ of R such that for each maximal ideal P of R the set of R_P-submodules of $D(R)_P$ containing R_P is linearly ordered. We then describe the distributive hull $D(R)$ in certain cases. In particular, we show that the distributive hull of a Noetherian integrally closed domain R is given by $\{\bigcap_{P \in X} R_P\} \cap K$, where X is the set of all maximal ideals of R of height greater than one and K is the field of quotients of R. If R is an Artinian ring, then $D(R) = R$. We also show that these results remain true when R is replaced by an ideal (restrictions may be imposed) of R.

Throughout R will denote a commutative ring with identity and MaxSpec R will denote the set of maximal ideals of R; if M is a submodule of an R-module N and $x, y \in N$, $(M : y) = \{r \in R \mid ry \in M\}$, $(x : y) = (Rx : y)$. If R is a ring, $Z(R)$ is the set of zero divisors of R.

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LEMMA 1.1. *Let R be a ring and $T = \{t \in R - Z(R) \mid Rt \subseteq R$ is distributive$\}$. Then T is a saturated multiplicatively closed subset of R.*

Proof. Let t_1 and t_2 be any two elements of T. Then clearly $t_1 t_2 \in R - Z(R)$. Since $Rt_1 \subseteq R$ is distributive, $(Rt_1 : r) + (r : s) = R$, for all

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$r \in R$ and all $s \in Rt_1$ [1, Proposition 1.1]. Since $t_2 \in R - Z(R)$, it follows that $(Rt_1t_2 : rt_2) + (rt_2 : st_2) = R$, for all $r \in R$ and all $s \in Rt_1$. Hence $Rt_1t_2 \subseteq Rt_2$ is distributive [1, Proposition 1.1]. But $Rt_2 \subseteq R$ is distributive. Therefore by [2, Lemma 2.1], $Rt_1t_2 \subseteq R$ is distributive. Hence $t_1t_2 \in T$.

Now let $u \in R$ and $t \in T$ such that $u \mid t$. Then $t = ru$, for some $r \in R - Z(R)$ and hence $Rru \subseteq R$ is distributive. Therefore it follows that $Rru \subseteq Rr$ is distributive. But then this implies that $Ru \subseteq R$ is distributive. Therefore $u \in T$.

The letter T defined in the above statement will retain the same meaning in the sequel.

COROLLARY. *Let R be a ring and S a multiplicatively closed subset of R contained in $R - Z(R)$. Then $R \subseteq S^{-1} R$ is distributive if, and only if, $S \subseteq T$.*

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In this section we state and prove our main results.

THEOREM 2.1. *Every ring R as a module over itself has a distributive hull $D(R)$, where $D(R)$ is a ring contained in the maximal quotient ring $Q(R)$ of R.*

Proof. Let R be a ring. Then $R_P \neq 0$, for all $P \in \operatorname{MaxSpec} R$. Therefore R has a maximal distributive extension [2, p.30]. To prove the uniqueness, we proceed as follows. Let $R \subseteq M$ be any distributive extension of R-modules. Then for any non-zero element m in M, we have $R = R \cap (Rm + R(1-m)) = (R \cap Rm) + (R \cap R(1-m))$. Hence $1 = am + b(1-m)$, for some $a, b \in R$ with $am \in R$ and $b(1-m) = b - bm \in R$; that is, $a, b \in (R : m)$. We now claim that $(R : m)\, m \neq 0$ and also $(R : m)\, r \neq 0$, for all non-zero elements r of R. Suppose that $(R : m)\, m = 0$. Then it follows that $am = bm = 0$, and hence $1 = am + b(1-m) = b$. *Thus* $1m = m = bm = 0$. This is a contradiction to the choice of m in M. Therefore $(R : m)\, m \neq 0$. Now for any non-zero element r of R, we have $r = ram + rb(1-m) \neq 0$. That is, $(R : m)\, r \neq 0$, for all non-zero elements r in R. Since m was taken to be any non-zero element of M, it follows that for all non-zero elements m in M, $(R : m)$ is a dense ideal of R [3, p. 37], and $(R : m)\, m \neq 0$. Hence by [3, Proposition 6, p. 40], M is isomorphic to a submodule of the maximal quotient ring $Q(R)$ of R. So by identifying M with its isomorphic copy in $Q(R)$, we have $R \subseteq M \subseteq Q(R)$.

Now take a maximal ideal P in R. Then $R_P \subseteq M_P \subseteq Q(R)_P$ and $R_P \subseteq M_P$ is a distributive extension of R_P-modules [2, Lemma 2.5.]. Hence for any $m/s \in M_P - R_P$, $R_P \subseteq R_P(m/s)$ [2, Lemma 2.7], which implies that

$1=(r/s')(m/s)$, for some $r/s' \in R_P$. Clearly r/s' and m/s are units in $Q(R)_P$ and $r/s' \in R_P - Z(R_P)$. We now set $T_P = \{t \in R_P - Z(R_P) \mid R_P t \subseteq R_P$ is distributive$\} \cap \{$units of $Q(R)_P\}$. Then $r/s' \in T_P$ and $m/s = s'/r \in T_P^{-1} R_P \subseteq Q(R)_P$. Since m/s was taken to be any element in $M_P - R_P$, it follows that $R_P \subseteq M_P \subseteq T_P^{-1} R_P \subseteq Q(R)_P$.

We now define $D(R) = \{x \in Q(R) \mid x/1 \in T_P^{-1} R_P$, for all $P \in \operatorname{MaxSpec} R\}$. Then what follows from the above argument is that $M \subseteq D(R)$ and that $D(R)$ contains an isomorphic image of every distributive extension of R. Since for each $P \in \operatorname{MaxSpec} R$, $R_P \subseteq D(R)_P \subseteq T_P^{-1} R_P$ in $Q(R)_P$, it follows that $R \subseteq D(R)$ is a distributive extension of R-modules [2, Lemma 2.6]. Hence $D(R)$ is the distributive hull of R.

To complete the proof we are left to show that $D(R)$ is a subring of $Q(R)$. For this take any two elements x_1, x_2 in $D(R)$. Then for all $P \in \operatorname{MaxSpec} R$, we have $x_1/1$, $x_2/1 \in T_P^{-1} R_P$. But $T_P^{-1} R_P$ is a ring. Therefore we have $(x_1 - x_2)/1$, $x_1 x_2/1$ are in $T_P^{-1} R_P$, for all $P \in \operatorname{MaxSpec} R$. Hence $x_1 - x_2, x_1 x_2 \in D(R)$. Therefore $D(R)$ is a subring of $Q(R)$.

Corollary 1 (cf. [2, Theorem 3.4]). *Let R be a local ring. Then $D(R) = T^{-1} R$.*

Corollary 2. *Let R be a ring with the distributive hull $D(R)$. Then for each maximal ideal P of R, the set of R_P-submodules of $(D(R)/R)_P$ is linearly ordered.*

Proof. Let P be a maximal ideal of R. Then $D(R)_P \subseteq T_P^{-1} R_P$. Therefore it is enough to show that for any two elements r_1/t_1, $r_2|t_2 \in T_P^{-1} R_P - R_P$, either $R_P(r_1/t_1) \subseteq R_P(r_2/t_2)$ or $R_p(r_2/t_2) \subseteq R_P(r_1/t_1)$. Now because $R_P \subseteq T_P^{-1} R_P$ is distributive, by [2, Lemma 2.7], we have $R_P \subseteq R_P(r_1/t_1)$ and $R_P \subseteq R_P(r_2/t_2)$, which implies that $R_P t_1 \subseteq R_P r_1$ and $R_P t_2 \subseteq R_P r_2$. Thus $t_1 = a r_1$ and $t_2 = b r_2$ for some a, $b \in R_P$. That is, $R_P(r_1/t_1) = R_P(1/a)$ and $R_P(r_2/t_2) = R_P(1/b)$. Since $R_P \subseteq R_P(1/a)$ and $R_P \subseteq R_P(1/b)$ are distributive extensions of R_P-modules, it follows that $R_P a \subseteq R_P$ and $R_P b \subseteq R_P$ are distributive extensions of R_P-modules. But then we have either $R_P a \subseteq R_P b$ or $R_P b \subseteq R_P a$. Therefore it follows that either $R_P(1/b) \subseteq R_P(1/a)$ or $R_P(1/a) \subseteq R_P(1/b)$. That is, either $R_P(r_2/t_2) \subseteq R_P(r_1/t_1)$ or $R_P(r_1/t_1) \subseteq R_P(r_2/t_2)$. The result now follows.

Because of its use we state the following.

Lemma 2.2. *Let R be a ring and $M \subseteq N$ an essential extension of R-modules. Suppose that $f\colon N \to X$ is an R-morphism of R-modules. If the induced map $f \mid M\colon M \to X$ is injective, then f is injective.*

Lemma 2.3. *Let R be a ring contained in a ring R'. If $R \subseteq R'$ is an essential extension of R-modules, then $Z(R) = Z(R') \cap R$.*

Proof. Let $a \in R$ and let $f_a: R' \to R'$ be defined by $f_a(x) = ax$, $x \in R'$. Then $a \in Z(R')$ if and only if f_a is not injective. Thus $f_a | R: R \to R$ has nonzero kernel. That is, $a \in Z(R)$, which proves that $R \cap Z(R') \subseteq Z(R)$. But clearly $Z(R) \subseteq R \cap Z(R')$. Therefore we have $Z(R) = R \cap Z(R')$.

Remark. Davison in [2, Proposition 3.5] claimed that for any ring R, the set $K(R) = \{x \in Q(R)$: for each maximal ideal P of R there is an element $s/1 \in R_P$ such that $R_P \subseteq R_P(1/s)$ is distributive an $(s/1)(x/1) \in R_P \subseteq Q(R)_P\}$ contains an isomorphic copy of every distributive ring extension of R and $R \subseteq K(R)$ is distributive. In claiming that $R \subseteq K(R)$ is distributive, he appears to assume that for each $P \in \operatorname{MaxSpec} R$, $R_P \subseteq Q(R)_P$ is an essential extension of R_P-modules. However, we have found no known proof of this. Since the subset $A = \{x \in Q(R)$: $R \subseteq R + Rx$ is not distributive and for each maximal ideal P of R there is an element $s/1$ of R_P such that $R_P \subseteq R_P(1/s)$ is distributive and $(s/1) \cdot (x/1) = 0\}$ of $Q(R)$ is contained in $K(R)$, there is a gap in Davison's proof that $R \subseteq K(R)$ is a distributive extension. However, there are cases where we have $K(R)$ is the distributive hull of R. We first recall that for any ring R, $R \subseteq Q(R)$ is an essential extension of R-modules.

PROPOSITION 2.4. *If either R is Noetherian or R is an integral domain, then $K(R)$ is the distributive hull of R.*

Proof. Since in either case $Q(R) = S^{-1}R$, where $S = R - Z(R)$ [4, Proposition 2.11, p. 204], it follows that $R_P \subseteq Q(R)_P$ is an essential extension of R_P-modules, for all $P \in \operatorname{MaxSpec} R$. Hence by Lemma 2.3, we have $R_P - Z(R_P) \subseteq Q(R)_P - Z(Q(R)_P)$. The result now follows.

PROPOSITION 2.5. *Let R be a ring and I be a regular ideal of R. Then every distributive extension of I has an isomorphic copy in the maximal quotient ring $Q(R)$ of R. If moreover I is locally invertible, then the submodule $D(I)$ of $Q(R)$ generated by the set of submodules of $Q(R)$ which are distributive extensions of I is the distributive hull of I and for each maximal ideal P of R, the set of R_P-submodules of $(D(I)/I)_P$ is linearly ordered.*

Proof. Let $I \subseteq M$ be any distributive extension of R-modules. Then $S^{-1}R = S^{-1}I \subseteq S^{-1}M$ is $S^{-1}R$ distributive $(S = R - Z(R))$ [2, Lemma 2.5]. Since $I \subseteq M$ is an essential extension of R-modules [2, Proposition 2.1], the mapping $m \to m/1$ of M into $S^{-1}M$ is an injection. Since $S^{-1}R \subseteq S^{-1}M$ is distributive, $S^{-1}M$ has an isomorphic copy in $Q(S^{-1}R)$ (Theorem 2.1). But $Q(S^{-1}R) = Q(R)$ [4, p. 205]. Therefore $Q(R)$ contains an isomorphic copy of M. The remaining part of the proof is similar to that of Theorem 2.1 and Corollary 2 to it.

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We recall that in a Noetherian local ring R with maximal ideal P, any non-zero distributive ideal is a power P^k of P.

LEMMA 3.1. *Let R be a Noetherian local integral domain with maximal ideal P. Suppose that there exists a non-zero ideal I of R such that $I \subsetneq P$ is a proper distributive extension of R-modules. Then R is of Krull dimension one.*

Proof. Let I be a non-zero ideal of R such that $I \subsetneq P$ is a distributive extension of R modules. Then $I = P^k = Pt$, for some $t \in P^{k-1}$ and some positive integer k [1, Proposition 1.2]. Since I is contained in the ideal generated by t, and t is not a unit, it follows that R is of Krull dimension one (Krull's principal ideal theorem).

PROPOSITION 3.2. *Let R be a Noetherian local ring with non-zero maximal ideal P. Then the following statements are equivalent.*

(i) $R \neq D(R)$

(ii) *R is a discrete valuation ring.*

Proof. (i) → (ii) $R \neq D(R) \subseteq S^{-1}R$ $(S = R - Z(R))$ implies that $Rt \subseteq R$ is distributive for some $t \in S \cap P$. So $Rt = P^k$ for some positive integer k. Hence P^k is an invertible ideal of R, which implies that P is an invertible ideal of R. Therefore R is a discrete valuation ring.

(ii) → (i) This is obvious.

PROPOSITION 3.3. *For any Artinian ring R we have $D(R) = R$.*

Proof. Since R is Artinian, R is Noetherian and so is a total quotient ring, so $R = D(R) = Q(R)$.

PROPOSITION 3.4. *Let R be an integral domain with field of fractions K. Then $D(R) = \bigcap_{P \in \mathrm{MaxSpec}\, R} (T_P^{-1} R_P) \subseteq K$, where $T_P = \{t \in R_P - \{0\} \mid R_P t \subseteq R_P$ is distributive$\}$.*

Proof. Let M be any distributive extension of R. Then M has an isomorphic copy in K (Proposition 2.4). So we may regard M as a submodule of K. Hence $R = \bigcap_{P \in \mathrm{MaxSpec}\, R} R_P \subseteq M = \bigcap_{P \in \mathrm{MaxSpec}\, R} M_P \subseteq K$. Since $T_P^{-1} R_P$ is the distributive hull of R_P in K, $R_P \subseteq M_P \subseteq T_P^{-1} R_P$ for all maximal ideals P of R. Therefore it follows that $R \subseteq M \subseteq \bigcap_{P \in \mathrm{MaxSpec}\, R} (T_P^{-1} R_P)$. Clearly $\bigcap_{P \in \mathrm{MaxSpec}\, R} (T_P^{-1} R_P) \subseteq T_Q^{-1} R_Q$ for any maximal ideal Q of R and $R_Q \subseteq (R_Q)_Q \subseteq (T_Q^{-1} R_Q)_Q$ is a distributive extension of R_Q-modules, for all maximal ideals Q of R. Therefore

$\bigcap_{P \in \operatorname{MaxSpec} R}(T_P^{-1}R_P)$ is a distributive extension of R and contains an isomorphic copy of every distributive extension of R. Hence $\bigcap_{P \in \operatorname{MaxSpec} R}(T_P^{-1}R_P)$ is the distributive hull of R.

When R is a Noetherian integral domain we can be more explicit about the expression of $D(R)$.

THEOREM 3.5. *Let R be a Noetherian integral domain. Then $D(R) = \bigcap_{P \in X} R_P$, where X consists of all the maximal ideals P of R such that R_P has no proper distributive extension at all. Moreover if R is integrally closed, then the set X consists of all the maximal ideals of R of height greater than one.*

Proof. By Proposition 3.4, $D(R) = \bigcap_{P \in \operatorname{MaxSpec} R}(T_P^{-1}R_P)$. Now if R_P has no proper distributive extension at all, then T_P is the set of units in R_P and hence $T_P^{-1}R_P = R_P$. If on the other hand R_P has a proper distributive extension, then $T_P^{-1}R_P = K$, the field of fractions of R (Proposition 3.2). Therefore it follows that $D(R) = \bigcap_{P \in X} R_P$.

If now R is assumed to be integrally closed, then the only case where $D(R_P) = K$ is that when P is of height one and for all the maximal ideals P of R of height greater than one, we have $D(R_P) = R_P$ [1, Proposition 4.5]. Therefore in this case X consists of all the maximal ideals of R of height greater than one.

THEOREM 3.6. *Let R be a Noetherian integrally closed domain and I an ideal of R. Then I has a distributive hull and this is given by $\bigcap_{P \in X} D(I_P)$, where X consists of all the maximal ideals of R of height greater than one and $D(I_P)$ is the distributive hull of R_P-module I_P.*

Proof. Since the case $I = 0$ is trivial, therefore we take I to be a non-zero ideal of R. Then clearly $I = \bigcap_{P \in \operatorname{MaxSpec} R} I_P$, and every distributive extension of I has an isomorphic copy in the field of quotients K of R (Proposition 2.5). Let P be any maximal ideal of R. Then we have two cases.

Case 1. Height of P is one. Then R_P is a discrete valuation ring and hence in this case we have $D(I_P) = K$.

Case 2. Height of P is greater than one. Suppose that I_P has a proper distributive extension. Then $I_P = P_P$ and $D(I_P) = R_P$ [1, Proposition 4.5]. Therefore as in the case of Proposition 3.4, we have $D(I) = \bigcap_{P \in X} D(I_P)$.

From the above proof it is clear that for each maximal ideal P of R, the set of submodules of $D(I_P)$ containing I_P is linearly ordered. Since for each maximal ideal P of R we have $D(I)_P \subseteq D(I_P)$, for each maximal ideal P of R the set of R_P-submodules of $D(I)_P$ containing I_P is linearly ordered.

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